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Some remarks on anti-de Sitter D-branes

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Abstract: We present some preliminary investigations about the $AdS_2 \times S^2$ D3-branes in $AdS_3 \times S^3$. We analyse the quadratic fluctuations of the Dirac-Born-Infeld action around a given semi-classical D-brane configuration and compare them with results obtained by using conformal-field-theory techniques. We finally study classical motions of open strings attached to those D-branes and analyse the rôle of the spectral flow in this context.

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Contents

1. Introduction and summary

Among the popular string backgrounds, anti-de Sitter spaces play an important rôle, both from phenomenological and fundamental viewpoints. They naturally arise in Randall-Sundrum “compactifications” as well as in near-horizon geometries of certain stringy black holes. The case of AdS_3 is even more intriguing, since together with a Neveu-Schwarz three-form, it provides an exact string background, which is described in terms of the $\mathrm{SL}_2(\mathbb{R})$ Wess-Zumino-Witten model [1, 2]. The latter can be embedded in a more general exact conformal field theory (CFT), $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SU}_2(\mathbb{C}) \times U(1)^4$. It is then automatically promoted to a critical type-IIB superstring [3, 4], which preserves 16 of the 32 type-II supercharges, and has $AdS_3 \times S^3 \times T^4$ as target space. This setting is precisely the near-horizon geometry of a black string, constructed out of Q_5 NS five-branes and Q_1 fundamental strings of type-IIB theory. Other three-dimensional anti-de Sitter black holes can be further obtained as orbifolds of the above CFT.

Despite many attempts, the resolution of the $\mathrm{SL}_2(\mathbb{R})$ WZW model is notoriously difficult, because the group manifold is neither compact, nor asymptotically flat. Progress on the perturbative closed-string spectrum, in particular, was made only recently [5, 6]. This exercise remains, however, challenging since, among others, it provides a unique setting in which to analyse the AdS/CFT correspondence beyond the supergravity approximation. Besides various phenomenological interests for studying the D-brane configurations in the $\mathrm{SL}_2(\mathbb{R})$ WZW model, the knowledge of their fluctuations is a valuable information for the string spectrum itself. Investigating this issue is one motivation of the present letter.

For compact Lie groups, the WZW D-branes are rather well understood [7]–[14] both from the CFT and from the geometric, target-space viewpoints. In fact, it has been shown in ref. [12] that the semi-classical analysis of D-branes for the group $SU_2(\mathbb{C})$ gives *exact* results for numerous CFT data. This is expected to work for more general groups, and is likely to be related to some underlying supersymmetry. For the non-compact group $SL_2(\mathbb{R})$, preliminary results were obtained in refs. [15, 16]. These results were extended in [17], where supersymmetric settings, in which the WZW D-branes for both $SL_2(\mathbb{R})$ and $SU_2(\mathbb{C})$ can be embedded, were studied in detail. The analysis, based on classical solutions of the low-energy (Dirac-Born-Infeld) action for the D-branes, reveals subtle features that are not present in the case of compact groups: unphysical brane trajectories, quantization conditions that are higher-order in the string coupling, divergent energies etc. A purpose of this note is to go beyond the classical analysis of [17], and further investigate the spectrum of fluctuations around the physical, supersymmetric brane solutions of $AdS_3 \times S^3$, namely the $AdS_2 \times S^2$ D3-branes. Whether our semi-classical results are exact is also discussed in some detail, but it remains an open question, which is not as easy to answer as it was in the pure $SU_2(\mathbb{C})$ WZW model.

We can summarize our results and the plan of the paper as follows. Section 2 is a reminder of the supersymmetric D-branes of $AdS_3 \times S^3$ and of the salient properties such as the origin of their stability related to charge quantization. In section 3, we compute quadratic fluctuations by expanding the Dirac-Born-Infeld (DBI) action around a given $AdS_2 \times S^2$ D3-brane. The spectrum, which is a combination of $SU_2(\mathbb{C})$ and (discrete) $SL_2(\mathbb{R})$ unitary representations, turns out to be independent of the particular D3-brane under consideration (i.e. independent of its electric and magnetic charges), as expected in perturbation theory. It is nevertheless compatible with the results obtained in section 4 by using a CFT analysis. This analysis is however incomplete because very little is known about $SL_2(\mathbb{R})$ Cardy boundary states and related open-string spectra. Consequently, the CFT techniques at hand can neither exhibit charge-dependent bounds on the permitted spins, nor identify the multiplicities of states constructed by spectral flow, in particular from continuous representations (density of long-string states). Notice that these representations and, more generally, the spectral-flow action are anyway invisible at the level of the DBI dynamics. In order to get more insight on those issues, one possibility is to turn back to the classical open string. This is done in section 5. After solving the type-D boundary conditions, we describe some specific solutions, which exhibit the behaviour of short and long strings, by using classical spectral flow to get the latter. The similarity with closed strings is striking in the case of the linear (i.e. electrically-neutral) D-brane, where all of spectral flow is preserved, whereas only half of it survives in the case of generic branes. This is linked to the reflection symmetry with respect to this D-brane, which also hints towards its open-string spectrum, and leads to an heuristic proof that the latter should be exactly "half" of the closed-string one.[1]

2. The $AdS_2 \times S^2$ D-brane in $AdS_3 \times S^3$: a reminder

We consider a type-IIB string on $AdS_3 \times S^3 \times T^4$. This is a supersymmetric background, where the radii of AdS_3 and S^3 are equal to each other, and it admits an exact CFT description as a supersymmetric WZW theory in $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SU}_2(\mathbb{C}) \times U(1)^4$. The levels of the $\mathrm{SL}_2(\mathbb{R})$ and $\mathrm{SU}_2(\mathbb{C})$ current algebras are, respectively k' and k, and are related to the radius[2] as follows:

$$L^2 = -(k'+2)\alpha' = (k+2)\alpha' \,. \tag{2.1}$$

Supersymmetry is expected to protect this equation, valid a priori in the weak-curvature (large-k) limit only, where k and $k+2$ are undistinguishable.

We parametrize the $AdS_3 \times S^3$ manifold with global coordinates such that the metric is

$$\begin{aligned} ds^2_{AdS_3} &= L^2\left[d\psi^2 + \cosh^2\psi\left(d\omega^2 - \cosh^2\omega\, d\tau^2\right)\right], \\ ds^2_{S^3} &= L^2\left[d\chi^2 + \sin^2\chi\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right], \end{aligned}$$

where $\psi, \omega, \tau \in \mathbb{R}$, $\chi, \theta \in [0,\pi]$ and $\phi \in [0, 2\pi]$. The Neveu-Schwarz two-form background reads, in a convenient gauge:

$$\begin{aligned} B_{AdS_3} &= L^2\left(\psi + \frac{\sinh 2\psi}{2}\right)\cosh\omega\, d\omega \wedge d\tau\,, \\ B_{S^3} &= L^2\left(\chi - \frac{\sin 2\chi}{2}\right)\sin\theta\, d\theta \wedge d\phi\,. \end{aligned}$$

The physical D3-branes that we will be analysing have $AdS_2 \times S^2$ geometry, and solve the classical equations of motion of the DBI action (see refs. [12, 17]), which reads:

$$S_{\mathrm{DBI}} = \int d^4\zeta\, \mathcal{L}_{\mathrm{DBI}}\,, \qquad \mathcal{L}_{\mathrm{DBI}} = -T\sqrt{-\det\left(\hat{g} + \hat{B} + 2\pi\alpha' F\right)}\,; \tag{2.2}$$

here $\hat{g}$ and $\hat{B}$ are the pull-backs of the WZW backgrounds, and F the world-volume U(1) field. The embedding of the $AdS_2 \times S^2$ branes under consideration in the $AdS_3 \times S^3$ ambient geometry is given by $\psi = \psi_0$, $\chi = \chi_0$. The AdS_2 part ($\psi = \psi_0$) looks like a static D-string stretching between two antipodal points lying on the

[1] Related considerations can be found in [19].

[2] The central charges are $c_{AdS_3} = 3k'/(k'+2)$ and $c_{S^3} = 3k/(k+2)$, and with these conventions $k' < -2$. By using (2.1), it is clear that those central charges add up to 6.

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boundary of AdS_3; for $\psi_0 = 0$, it is just a straight line passing through the center of AdS_3 (which we call *linear* D-brane). The induced radii are $\ell_{AdS_2} = L\cosh\psi_0$ and $\ell_{S^2} = L\sin\chi_0$. The natural world-volume coordinates are ω, τ, θ and ϕ, and the covariantly-constant U(1) field carried by those D3-branes, in the above gauge, takes the form:

$$F_0 = dA_0 = -\frac{L^2}{2\pi\alpha'}\left(\psi_0 \cosh\omega\, d\omega \wedge d\tau + \chi_0 \sin\theta\, d\theta \wedge d\phi\right). \tag{2.3}$$

Both factors in $AdS_2 \times S^2$ are generalized (twined) conjugacy classes of the groups $\mathrm{SL}_2(\mathbb{R})$ and $\mathrm{SU}_2(\mathbb{C})$, invariant under a "diagonal" $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SU}_2(\mathbb{C})$ part of the original isometry group. This defines a natural $s\ell_2(\mathbb{R})\times \mathrm{su}_2(\mathbb{C})$ action on the functions on this D3-brane, whose quadratic Casimirs are the d'Alembert operators, normalized to unit radii:

$$\begin{aligned}
\Box_{AdS_2} &= -\frac{1}{\cosh^2\omega}\partial_\tau^2 + \frac{1}{\cosh\omega}\partial_\omega \cosh\omega\partial_\omega\,, \\
\Box_{S^2} &= -\triangle_{S^2} = -\frac{1}{\sin^2\theta}\partial_\phi^2 - \frac{1}{\sin\theta}\partial_\theta \sin\theta\partial_\theta\,.
\end{aligned}$$

A "mini-superspace" analysis of (2.2), taking into account the degrees of freedom corresponding only to rigid motions of the $AdS_2 \times S^2$ D3-brane in the ψ and χ directions, shows that the constants ψ_0 and χ_0 are quantized. The origin of their quantization is, however, different. On the one hand, for the $\mathrm{SU}_2(\mathbb{C})$ component, χ_0 is discrete as a consequence of the quantization of the magnetic flux through S^2 [12], which ensures the stability of the brane against shrinking to zero size: $\chi_0 = \pi\alpha' p/L^2$, with p integer. On the other hand, for the $\mathrm{SL}_2(\mathbb{R})$ part, one can advocate that the Wilson line around a closed string, whose momentum is the electric charge, is cyclic; as was argued in [17], locality demands that the quantization be more generally valid, and hence applicable to the case of AdS_2 branes: $\psi_0 = \mathrm{arcsinh}(q/2\pi\alpha' p T_{\mathrm{D}})$ (T_{D} is the D-string tension). We will comment shortly on this issue in what follows. Notice, finally, that χ_0 is bounded and so is the magnetic charge p, while ψ_0 and consequently the (integer) electric charge q are not.

3. Quadratic fluctuations around $AdS_2 \times S^2$

Our aim is now to go further, and derive the complete spectrum of small quadratic fluctuations around these D-brane solutions. Those fluctuations are captured in the following degrees of freedom:

$$\chi = \chi_0 + \delta\chi\,, \qquad \psi = \psi_0 + \delta\psi\,, \qquad \text{and} \qquad A = A_0 + \delta A\,, \tag{3.1}$$

where $F_0 = dA_0$ is given in eq. (2.3), while $\delta\chi$, $\delta\psi$ and δA are arbitrary functions of the four world-volume coordinates, supplemented with some gauge condition. We ignore the fluctuations of the brane along the T^4.

In plugging (3.1) in (2.2) and expanding out to quadratic order, we observe that the final expression starts with linear terms. This is not in contradiction with our assertion that we are expanding around a classical solution: the quantization of electric and magnetic charges, discussed above, forces these terms to be set to zero. The S^2 part, which is proportional to $\int d\phi\, d\theta\, \delta F_{\phi\theta}$, should be dropped because the magnetic flux is quantized . The AdS_2 part is proportional to $\int d\omega\, d\tau\, \delta F_{\omega\tau}$. But unlike S^2, AdS_2 is topologically trivial so this is bound to be zero if nothing evil lingers at infinity. We will soon be forced to make a similar assumption in order to compute the second-order terms. We are allowed to impose boundary conditions such that $\int d\omega\, d\tau\, \delta F_{\omega\tau} = 0$ and $\delta\psi \to 0$ thanks to the quantized AdS_2 charge $q = (1/4\pi L^2) \int_{S^2} dv_2 \frac{\partial \mathcal{L}_{\rm DBI}}{\partial\, \partial_\varphi A_\tau}$ (with φ the angular coordinate of AdS_3 in cylindrical coordinates and dv_2 the normalized two-sphere volume form). This rules out the constant modes of $\delta\psi$ and $\delta F_{\omega\tau}/\cosh\omega$, which are linked by the equations of motion as we will see later.

We now turn to the second-order terms — the fluctuation of the action, which, up to an irrelevant multiplicative constant, read:

$$
\begin{aligned}
\delta^{(2)} S_{\rm DBI} \propto \int d\omega\, d\tau\, d\theta\, d\phi \frac{\cosh\omega}{\sin\theta} & \left[\sin^2\theta(\partial_\theta\delta\psi)^2 + (\partial_\phi\delta\psi)^2 + \sin^2\theta(\partial_\theta\delta\chi)^2 + (\partial_\phi\delta\chi)^2 + \right. \\
& \left. + 2\sin^2\theta(\delta\chi)^2 + (\delta F_{\phi\theta})^2 + 4\sin\theta\, \delta\chi\, \delta F_{\phi\theta}\right] + \\
+ \frac{\sin\theta}{\cosh\omega} & \left[\cosh^2\omega(\partial_\omega\delta\chi)^2 - (\partial_\tau\delta\chi)^2 + \cosh^2\omega(\partial_\omega\delta\psi)^2 - (\partial_\tau\delta\psi)^2 - \right. \\
& \left. - 2\cosh^2\omega(\delta\psi)^2 - (\delta F_{\tau\omega})^2 - 4\cosh\omega\, \delta\psi\, \delta F_{\tau\omega}\right] + \\
+ & \left[\frac{\cosh\omega}{\sin\theta}(\delta F_{\omega\phi})^2 + \cosh\omega\sin\theta(\delta F_{\omega\theta})^2 - \frac{1}{\cosh\omega\sin\theta}(\delta F_{\tau\phi})^2 - \right. \\
& \left. - \frac{\sin\theta}{\cosh\omega}(\delta F_{\tau\theta})^2\right].
\end{aligned}
$$

In order to obtain such a concise result, we have performed some partial integrations. A sufficient assumption is that δA and its derivatives be continuous functions on $AdS_2 \times S^2$ that vanish at the infinity of AdS_2. Of course, our expression does not depend on the position of the D-brane, i.e. on the conserved charges p and q; this is a general feature of (twined-) conjugacy-class D-branes in group manifolds.

The linearized equations of motion can be derived from $\delta^{(2)} S_{\rm DBI}$:

$$
\begin{bmatrix}
\Box_{AdS_2} - \Box_{S^2} + 2 & 2 & & \\
2\Box_{AdS_2} & \Box_{AdS_2} - \Box_{S^2} & \multicolumn{2}{c}{\mathbf{0}} \\
 & & \Box_{AdS_2} - \Box_{S^2} - 2 & -2 \\
\multicolumn{2}{c}{\mathbf{0}} & -2\Box_{S^2} & \Box_{AdS_2} - \Box_{S^2}
\end{bmatrix}
\begin{bmatrix}
\delta\psi \\ \frac{\delta F_{\omega\tau}}{\cosh\omega} \\ \delta\chi \\ \frac{\delta F_{\theta\phi}}{\sin\theta}
\end{bmatrix} = 0\,,
$$

$$
\begin{aligned}
\Box_{AdS_2}\left(-\frac{1}{\sin^2\theta}\partial_\phi\delta A_\phi - \frac{1}{\sin\theta}\partial_\theta \sin\theta\, \delta A_\theta\right) - \\
- \Box_{S^2}\left(-\frac{1}{\cosh^2\omega}\partial_\tau\delta A_\tau + \frac{1}{\cosh\omega}\partial_\omega \cosh\omega\, \delta A_\omega\right) = 0\,;
\end{aligned}
$$

these may be supplemented by the following covariant gauge condition:

$$\frac{1}{\cosh^2\omega}\partial_\tau \delta A_\tau - \frac{1}{\cosh\omega}\partial_\omega \cosh\omega\, \delta A_\omega - \frac{1}{\sin^2\theta}\partial_\phi \delta A_\phi - \frac{1}{\sin\theta}\partial_\theta \sin\theta\, \delta A_\theta = 0\,. \qquad (3.2)$$

That this condition does not leave any spurious degrees of freedom will become clear here after, when comparing these results with the CFT approach.

Now, let us solve these equations, while diagonalizing the operators $\Box_{AdS_2}$ and $\Box_{S^2}$. We may first look for solutions such that

$$\delta\chi = \delta A_\phi = \delta A_\theta = -\frac{1}{\cosh^2\omega}\partial_\tau \delta A_\tau + \frac{1}{\cosh\omega}\partial_\omega \cosh\omega \delta A_\omega = 0\,. \qquad (3.3)$$

Then if we write $\Box_{S^2} = j(j+1)$ we must have $\Box_{AdS_2} = j(j-1)$ or $\Box_{AdS_2} = (j+1)(j+2)$. Let us be more explicit on the first possibility. Given a function f on the brane such that $\Box_{S^2} f = j(j+1)f$ and $\Box_{AdS_2} f = j(j-1)f$, we have a solution of the form:

$$\delta\psi = f\,, \qquad \frac{\delta F_{\omega\tau}}{\cosh\omega} = (j-1)f\,, \qquad \delta A_\tau = \frac{1}{j}\cosh\omega\partial_\omega f\,, \qquad \delta A_\omega = \frac{1}{j}\frac{\partial_\tau f}{\cosh\omega}\,.$$

We observe that from the case $j = 0$, we learn that the constant mode of $\delta\psi$ corresponds to a constant mode of $\delta F_{\omega\tau}/\cosh\omega$, as advertised previously. As the latter is excluded by charge conservation, the former should also be excluded from the spectrum. Similar solutions are obtained with a function g on the brane such that $\Box_{S^2} g = j(j+1)g$ and $\Box_{AdS_2} g = (j+1)(j+2)g$, with $j \geq 0$. Therefore, the full set of solutions compatible with (3.3) spans the following representations of $\mathrm{SU}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{R})$:

$$\bigoplus_{j\in\mathbb{Z}, j\geq 1} (j, j-1) \oplus \bigoplus_{j\in\mathbb{Z}, j\geq 0} (j, j+1)\,. \qquad (3.4)$$

Note that starting from a spin-j representation of $\mathrm{SU}_2(\mathbb{C})$ (with j a positive integer), our equations only allow positive-Casimir integer-spin representations of $\mathrm{SL}_2(\mathbb{R})$. These are *discrete*[3] *representations*, which have a lowest- or highest-weight state and infinite dimension. We will return to this issue in the next section.

Instead of (3.3) one could equivalently assume that

$$\delta\psi = \delta A_\omega = \delta A_\tau = -\frac{1}{\sin^2\theta}\partial_\phi \delta A_\phi - \frac{1}{\sin\theta}\partial_\theta \sin\theta \delta A_\theta = 0\,.$$

The decomposition in representations follows the same pattern as before, leading thereby again to (3.4).

[3]For concreteness, we remind that general discrete lowest- (highest-)weight representations of $\mathrm{SL}_2(\mathbb{R})$, $\mathcal{D}_j^\pm$, are spanned by $\{|j,m\rangle, m = \pm(j+1), \pm(j+2), \ldots\}$ and are unitary for real $j \geq -1$. Their Casimir, $j(j+1)$, is positive for positive j. Continuous representations have negative Casimir, parametrized by $j = -1/2 + is$.

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Finally, the last class of solutions we must consider obeys

$$\delta\chi = \delta\psi = \delta F_{\omega\tau} = \delta F_{\phi\theta} = 0\,.$$

For these

$$\delta A_i = \partial_i h\,,$$

where h is any function on the brane. Now, the gauge condition (3.2) reads: $\Box_{AdS_2} h = \Box_{S^2} h$. This restricts the allowed solutions to the following members of the $\mathrm{SU}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{R})$ representations:

$$\bigoplus_{j\in\mathbb{Z}, j\geq 1} (j, j)\,.$$

To summarize, the $\mathrm{SU}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{R})$ content of the solutions of the first-order Dirac-Born-Infeld equations of motion (plus gauge condition) is

$$2(0,1) \oplus \bigoplus_{j\in\mathbb{Z}, j\geq 1} (j, 2(j-1) \oplus j \oplus 2(j+1))\,. \tag{3.5}$$

It is worthwhile to stress that no upper bound on j steams out of the present semi-classical analysis.

4. Open-string states: towards a CFT analysis

The alternative description of D-branes is through CFT on the world-sheet. The set of semi-classical D3-brane configurations, $AdS_2 \times S^2$, considered here should be identified with the Cardy boundary states, preserving an $\mathrm{SU}_2(\mathbb{C})\times\mathrm{SL}_2(\mathbb{R})$ symmetry. As far as the spectrum of quadratic fluctuations is concerned, it should be compared with the open-string excitations of the corresponding Hilbert space.

Let us consider the $\mathrm{SU}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{R})$ CFT,[4] whose operators are the $\mathrm{SU}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{R})$ currents $J^A(z)$. We will write highest-weight states of this theory according to their behaviour under the zero-mode $\mathrm{SU}_2(\mathbb{C})\times\mathrm{SL}_2(\mathbb{R})$ subalgebra of the $\mathrm{SU}_2(\mathbb{C})\times\mathrm{SL}_2(\mathbb{R})$ current algebra: $|j, j'\rangle$, where j (resp. j') is a representation of $\mathrm{SU}_2(\mathbb{C})$ (resp. $\mathrm{SL}_2(\mathbb{R})$) of quadratic Casimir $j(j+1)$ (resp. $j'(j'+1)$). To construct the other states we have to apply modes of the J^As to these.

As already mentioned, the $AdS_3\times S^3$ background is such that AdS_3 and S^3 have equal radii, so that in the corresponding WZW conformal model the current algebras associated to the groups $\mathrm{SU}_2(\mathbb{C})$ and $\mathrm{SL}_2(\mathbb{R})$ have levels related according to eq. (2.1). This ensures that the mass-shell condition[5] for the lightest (level-one) states is just

[4] The superconformal field theory works the same way for our purposes.

[5] We neglect the T^4 contribution to the conformal weights since we have ignored the fluctuations of the D-brane in those directions.

$j(j+1)-j'(j'+1)=0$, which implies $j=j'$ with our conventions. These open-string states are obtained by acting on highest-weight states $|j,j\rangle$ with the operators J^A_{-1}. Thus the $\mathrm{SU}_2(\mathbb{C})\times\mathrm{SL}_2(\mathbb{R})$ content of this set of states is[6]

$$\bigoplus_{j\in\mathbb{Z},j\geq 0}[(1,0)+(0,1)]\otimes|j,j\rangle = 2|0,1\rangle\oplus\bigoplus_{j\in\mathbb{Z},j\geq 1}2|j,(j-1)\oplus j\oplus(j+1)\rangle\,. \quad (4.1)$$

Now, we must impose the Virasoro L_1 condition. If we note Adj the adjoint representation of the group G (here $\mathrm{SU}_2(\mathbb{C})\times\mathrm{SL}_2(\mathbb{R})$), which is realized by the J^A_{-1}s, and Rep any representation, then the Virasoro L_1 constraint is a G-covariant map $\mathrm{Adj}\otimes\mathrm{Rep}\to\mathrm{Rep}$. Hence any irreducible subrepresentation of $\mathrm{Adj}\otimes\mathrm{Rep}$ which does not appear in Rep lies in its kernel.

Thus the Virasoro constraint eliminates one of the two $|j,j\rangle$ states and we are left with the same representations that were found in the previous section, eq. (3.5).

Several remarks are in order here. As we already observed in section 3, when analysing the quadratic fluctuations of the D3-brane, the presence of $\mathrm{SU}_2(\mathbb{C})$ imposes a selection rule for the allowed $\mathrm{SL}_2(\mathbb{R})$ states: only discrete integer- and positive-spin (i.e. positive-Casimir) representations appear, without any bound on the allowed (positive) spin (see (3.5)). There are several reasons to believe that the agreement with the above CFT analysis is at most partial. Firstly, in eq. (4.1), the spin j is necessarily bounded, as a consequence of the unitarity of the $\mathrm{SU}_2(\mathbb{C})$ current-algebra representations: $j\leq k/2$. This bound is expected to be even stronger, and related to the magnetic charge of the Cardy state describing the D-brane under consideration. Secondly, our CFT analysis was minimal. We considered highest-weight representations of the $\mathrm{SL}_2(\mathbb{R})$ current algebra based on discrete representations of the zero-mode, global algebra. It is clear, however, that other (non-highest-weight) representations must be taken into account, which are obtained by spectral flow. These are built on both discrete and continuous representations of $\mathrm{SL}_2(\mathbb{R})$, and allow for a systematic construction of the closed-string spectrum (short-string versus long-string states) [5]. However, the spectrum of quadratic fluctuations of the DBI action around a semi-classical D-brane configuration is due to be blind to the effect of the spectral flow, the latter being a symmetry of the current algebra. In that respect, eq. (3.5) deals only with a part of the complete open-string spectrum. In order to understand these issues and determine the multiplicities of representations, a more complete analysis is needed, which should rely on the knowledge of the spectrum of Cardy boundary states and of the exact open-string partition function. Those data are not available at present, and their determination is beyond the scope of this

[6]Note that the tensor product of $\mathrm{SL}_2(\mathbb{R})$ representations is not as standard as the $\mathrm{SU}_2(\mathbb{C})$ case since we are dealing with infinite-dimensional representations. Anyway, the traditional spin composition rule holds for tensor products between one finite-dimensional representation of spin j and one highest- or lowest-weight representation of spin greater than j.

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work.[7] Instead, studying classical motions of an open string attached to a D-brane, can provide helpful information in that direction and clarify, in particular, the rôle of the spectral flow.

5. Classical open strings in AdS_3

In the following, we restrict our attention to open strings in AdS_3 ending on an AdS_2 D-brane and ignore the S^2 factor (we also ignore factors of k' and α', which have no influence for our purpose). The world-sheet dynamics is described by the WZW action with appropriate boundary conditions. Let $x^\pm = \sigma^0 \pm \sigma^1$ be world-sheet coordinates ($\sigma^1 \in [0, \pi]$) and $g(x^\pm)$ the embedding of the bosonic open string in $\mathrm{SL}_2(\mathbb{R})$. Let $J_+ = -\partial_+ g g^{-1}$ and $J_- = g^{-1}\partial_- g$ be the world-sheet currents. WZW equations of motion thus read: $\partial_\mp J_\pm = 0$. Let ω be the outer automorphism of the group $\mathrm{SL}_2(\mathbb{R})$, i.e. the conjugation by the matrix $\Omega = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ (we note ${}^\omega h = \Omega h \Omega^{-1}$ its action on any matrix h of $\mathrm{SL}_2(\mathbb{R})$ or $s\ell_2(\mathbb{R})$). The four type-D (type-N) untwisted (twisted) boundary conditions (see for example [10]) are $J_- = (-)\,{}^{(\omega)}J_+$.

We will first determine which of these four possible boundary conditions are physical. We consider the tangent vector to the world-sheet along the σ^0 direction. Its norm reads:

$$N = g_{\mu\nu}\partial_{\sigma^0}X^\mu \partial_{\sigma^0}X^\nu = \mathrm{Tr}\left(g^{-1}\partial_{\sigma^0} g\right)^2 = \frac{1}{4}\,\mathrm{Tr}\left(g^{-1}J_- g - J_+\right)^2 .$$

The Virasoro constraint $T_{++} = \mathrm{Tr}\, J_+^2 < 0$ (we assume some internal unitary CFT giving a positive contribution to the stress tensor) and the boundary conditions enable us to determine the sign of N. More precisely, in the cases of untwisted type-N and twisted type-D (AdS_2 D-brane), we have $N < 0$ (not only on the boundary). Therefore, the AdS_2 D-brane is physical. On the contrary, the twisted type-N and untwisted type-D (dS_2 D-brane) do not forbid $N > 0$, and lead to $g_{\mu\nu}\partial_{\sigma^1}X^\mu\partial_{\sigma^1}X^\nu < 0$, so they are unphysical. This has already been argued in ref. [17] by observing that the electric field is supercritical on the dS_2 D-brane. The contact with the present discussion is made by noting that the electric field is related to the boundary conditions (see for instance [18]), which can be written in terms of the tangent vector to the string at its end-points.

Let us focus now on the open-string solutions in the case of our D-brane. The solutions of the bulk equations are $g = a(x^+)\, b(x^-)$, with $a(0) = 1$ to ensure the uniqueness of a and b. We must impose boundary conditions, which are equivalent to $\partial_{\sigma^0}(b\, {}^\omega a) = 0$ on the boundary (the type-N boundary conditions cannot be solved

[7]Such data are accessible via computation of the emission amplitude of a closed string off the brane (cylinder diagram). Among various difficulties, we expect severe divergence problems due to the AdS_3 geometry.

so easily). Let us introduce

$$\begin{aligned} m &= b\,{}^{\omega}a\ (\sigma^1 = 0) = b(\sigma^0)\,{}^{\omega}a(\sigma^0)\,, \\ \bar{m} &= b\,{}^{\omega}a\ (\sigma^1 = \pi) = b(\sigma^0 - \pi)\,{}^{\omega}a(\sigma^0 + \pi)\,. \end{aligned} \tag{5.1}$$

These define the twined conjugacy classes to which the $\sigma^1 = 0$ and $\sigma^1 = \pi$ ends of the string belong, respectively, and allow us to write b in terms of a. Then the solution takes the form:

$$g = a(x^+)\, m\, ({}^{\omega}a(x^-))^{-1}\,, \tag{5.2}$$

where the $\bar{m}$ condition (5.1) remains to be implemented.

This expression holds actually in a general group manifold, but is not very visual even in the simple case of our AdS_2 stretched static D-string in AdS_3. We will thus restrict our attention to some special solutions such that J_+ is a constant. This is a generalization of the procedure of [5] for constructing non-trivial closed-string solutions by spectral-flowing constant-J_+ ones (which are in that case geodesics viewed as degenerate strings). Equation (5.2) now reads:

$$g = e^{x^+ C} m\, e^{-x^-\, {}^{\omega}C}\,, \tag{5.3}$$

with C an $s\ell(2)$ matrix such that $\mathrm{Tr}(\, e^{2\pi C} - 1) m\,\Omega = 0$ (this condition means that m and $\bar{m}$ belong to the same twined conjugacy class).

If $\det C > 0$ we have various sights according to the value of $r = \sqrt{\det C}$. In all cases the solutions are time-like and stay at finite distance from the origin; τ is a monotonous function of σ^0 and the other coordinates are periodic: the solutions should be considered as *short strings*. Both ends of the string go from $\tau = -\infty$ to $\tau = +\infty$ when σ^0 varies in the same range.[8] At a given world-sheet time σ^0, we see the string turn with an angle roughly equal to $2\pi r$ around the center of AdS_3; the integer part of r counts the number of complete revolutions around this point.

If $\det C = 0$, the string world-sheet degenerates into a light-like geodesic on the D-brane. This geodesic goes from one boundary of AdS_2 to the other in finite time τ. At a given σ^0 the string spans a segment of the geodesic (a point if $\psi_0 = 0$).

If $\det C < 0$ the solution (5.3) is spurious: the tangent vector ∂_{σ^0} is space-like. This is easier to visualize in the special case of the linear D-brane ($\psi_0 = 0$), where the open string world-sheet degenerates into a line which is a space-like geodesic. In the general case the string moves from a point of one boundary (say $\omega = -\infty$) of AdS_2 when $\sigma^0 = -\infty$ to a point of the other boundary when $\sigma^0 = +\infty$. It remains on the side of the D-brane where the center of AdS_3 is.

The spurious character of the last solution is due to the positivity of its T_{++}. To remedy this, we are led to make use of the spectral-flow symmetry that we mentioned earlier. This symmetry of the $\mathrm{SL}_2(\mathbb{R})$ WZW-model holds for open-string solutions in

[8] To interpret the $\mathrm{SL}_2(\mathbb{R})$ matrix g of (5.3) as an element of AdS_3, we have to decompactify τ.

the following way: take the general open-string solution in AdS_3 with twisted type-D boundary conditions and both ends of the string on the same D-brane $\psi = \psi_0$. Such a solution is defined by $a(x)$ and m, eq. (5.2). Then another solution is

$$\tilde{a}(x) = \exp\left(\frac{i}{2} w x \sigma_2\right) a(x) ,$$

with σ_2 the standard Pauli matrix. The points $\sigma^1 = \pi$ will remain on the same D-brane if w is an even integer. If w is an odd integer they move to the opposite D-brane at $\psi = -\psi_0$. So in general *half of the spectral-flow symmetry is preserved by the D-brane.*

The spectral flow changes T_{++} in the same way as in the closed-string case. Thus it can change its sign. In the case $\det C < 0$, it results in *long-string-like solutions* which approach the boundary of AdS_3 as σ^0 and τ go to infinity, with both ends of the string coming closer one from another and from the boundary. When $\det C > 0$, the transformation $r \to r + 1$ can be interpreted as non-standard spectral flow using the matrix C/r instead of $i\sigma_2$ (this is allowed because C/r also has eigenvalues i and $-i$ so that $\exp(2\pi C/r) = 1$).

We finally concentrate on the electrically-neutral D-brane $\psi_0 = 0$, which is a straight line stretching between two antipodal points of the boundary of AdS_3 and passing through its center. The corresponding open-string model can be viewed as "half of the closed-string one". More precisely, the twisted type-D boundary conditions are of the familiar form

$$\psi = 0 , \qquad \partial_{\sigma^1}\tau = 0 , \qquad \partial_{\sigma^1}\omega = 0 ,$$

and any classical open-string solution ending on the D-brane $\psi_0 = 0$ can be extended to give a closed-string one defined for $\sigma^1 \in [-\pi, 0]$ by:

$$\psi(\sigma^1) = -\psi(-\sigma^1) , \qquad \tau(\sigma^1) = \tau(-\sigma^1) , \qquad \omega(\sigma^1) = \omega(-\sigma^1) . \tag{5.4}$$

Of course, any closed-string solution obeying this symmetry can also be interpreted as an open-string one. Thus, in this case, the spectral-flow parameter w is not restricted to be even. This reflection symmetry should also hold at the level of the spectrum. That is, we expect not only a matching of classical excitations but also of CFT states of the open- and closed-string theories. More, there should also be a matching of multiplicites of those states (not only of zero multiplicities). The conclusion of these heuristic arguments is that *the open-string spectrum of the D-brane $\psi_0 = 0$ is given by exactly "half" of the closed-string spectrum* (i.e. the chiral CFT spectrum that should be tensored with itself and level-matched in order to generate the closed-string spectrum). If the latter is as proposed in [5], then the former will be made of the following representations of the $\mathrm{SL}_2(\mathbb{R})$ current algebra: $\hat{\mathcal{C}}^{\alpha,w}_{-1/2+is}$, $\hat{\mathcal{D}}^{+,w}_{j'}$, with the same range $j' \in]\frac{1}{2}, \frac{k'-1}{2}[$ (and the same weight for the continuous representations).

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